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Unicity in One-Sided L_1-Approximation and Quadrature Formulae

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It is shown that for each n-dimensional subspace G of $C(T)$ which contains a strictly positive function there exists a quadrature formula with at most $n-1$ points and positive weights which is exact for all $g \in G$ ($n \geqslant 2$, T has at most $n-1$ components). Since unicity of best one-sided L_1-approximations from G is equivalent to the non-existence of such quadrature formulae, a general non-unicity theorem is obtained. This result does not hold if T has more than $n-1$ components.

INTRODUCTION

The connection of unicity in one-sided L_1-approximation with the existence of quadrature formulae is investigated. Let an n-dimensional subspace G of $C(T)$ (T a compact metric space) which contains a strictly positive function and $f \in C(T)$ be given. A function $g_f \in G$ with $g_f \leqslant f$ is called a best one-sided L_1-approximation of f if $\|f-g_f\|_1 \leqslant \|f-g\|_1$ for all $g \in G$ with $g \leqslant f$.

Applying a characterization on unique solutions of semi-infinite optimization problems given in [7] to this approximation problem, the following characterization is obtained: Every $f \in C(T)$ has a unique (resp. strongly unique) best one-sided L_1-approximation from G if and only if there does not exist a quadrature formula of at most $n-1$ points with positive weights which is exact for all $g \in G$.

We show that if $n \geqslant 2$ and T has at most $n-1$ components, then for each G there exists such a quadrature formula. By applying the above characterization we obtain a general non-unicity result on one-sided L_1-approximation. Special cases of this theorem were proved for $T=[a, b]$ and under the additional assumption that G is a Haar subspace by DeVore

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[2], a subspace of splines by Pinkus [9], and a weak Chebyshev subspace by Strauss [11].

Finally, it is shown that the above results do not hold if T has more than $n-1$ components.

The main results in this paper have been presented in a talk at the GAMM meeting in 1984. An abstract of this talk is given in [6].

1. Preliminaries

We first recall a result on global unicity in semi-infinite optimization given in [7] which will be applied to one-sided L_1-approximation. To do this we need the following notation.

For a compact metric space we denote by $C(T)$ (resp. $C(T, \mathbb{R}^n)$) the space af all the continuous mappings $h: T \to \mathbb{R}$ (resp. $B: T \to \mathbb{R}^n$). Let $p \in \mathbb{R}^n$, $B \in C(T, \mathbb{R}^n)$, $f \in C(T)$, and $\sigma = (p, B, f)$ be given. We consider the corresponding linear *semi-infinite optimization problem* $LM(\sigma)$.

$$\text{Minimize } \langle p, x \rangle = \sum_{i=1}^{n} p_i x_i \text{ subject to}$$

$$\langle B(t), x \rangle \leqslant f(t) \text{ for all } t \in T.$$

The optimization problem $LM(\sigma)$ is said to satisfy the *Slater-condition* if there exists a vector $y \in \mathbb{R}^n$ such that

$$\langle B(t), y \rangle < f(t) \qquad \text{for all } t \in T.$$

We set $L = \{\sigma: LM(\sigma)$ has a solution and satisfies the Slater-condition$\}$. A solution $x \in \mathbb{R}^n$ of $LM(\sigma)$ is called *strongly unique* if there exists a constant $K > 0$ such that for all feasible points $y \in \mathbb{R}^n$ (i.e., $\langle B(t), y \rangle \leqslant f(t)$ for all $t \in T$),

$$\langle p, y \rangle \geqslant \langle p, x \rangle + K \|x - y\|_2.$$

The following unicity result was given in [7].

THEOREM 1. *For a fixed vector $p \in \mathbb{R}^n$, $p \neq 0$, and a fixed mapping $B \in C(T, \mathbb{R}^n)$, $B \neq 0$, the following statements are equivalent:*

(1) *For every $f \in C(T)$ with $\sigma = (p, B, f) \in L$ the optimization problem $LM(\sigma)$ has a unique solution.*

(2) *For every $f \in C(T)$ with $\sigma = (p, B, f) \in L$ the optimization problem $LM(\sigma)$ has a strongly unique solution.*

(3) *There do not exist points* $t_1, \ldots, t_{n+1} \in T$ *and real numbers* $a_1, \ldots, a_{n+1} \geqslant 0$ *such that*

$$-p = \sum_{i=1}^{n+1} a_i B(t_i).$$

It is well known that one-sided L_1-approximation is a special semi-infinite optimization problem.

Let μ be a strictly positive measure on T (i.e., $\int_T f d\mu > 0$ for all $f \in C(T)$ with $f \geqslant 0$ and $f \neq 0$). Furthermore, let the L_1-norm of a function $f \in C(T)$ be defined by

$$\| f \|_1 = \int_T | f | \, d\mu.$$

Now, let $G = \operatorname{span}\{g_1, \ldots, g_n\}$ be an n-dimensional subspace of $C(T)$ and $f \in C(T)$. The *one-sided* L_1-*approximation problem* is to find a function $g_f \in G$ with $g_f \leqslant f$, called *best one-sided* L_1-*approximation* of f from G, such that for all $g \in G$ with $g \leqslant f$,

$$\| f - g \|_1 \geqslant \| f - g_f \|_1.$$

A function $g_f \in G$ with $g_f \leqslant f$ is called a *strongly unique best one-sided* L_1-*approximation* of f from G, if there exists a constant $K > 0$ such that for all $g \in G$ with $g \leqslant f$,

$$\| f - g \|_1 \geqslant \| f - g_f \|_1 + K \| g - g_f \|_1.$$

It is easy to see that this approximation problem can be written as an optimization problem $LM(\sigma)$, where

$$p = \left(-\int_T g_1 \, d\mu, \ldots, -\int_T g_n \, d\mu \right)$$

and

$$B(t) = (g_1(t), \ldots, g_n(t)) \qquad \text{for all } t \in T.$$

Obviously, if G contains a strictly positive function, then $LM(\sigma)$ satisfies the Slater-condition for all $f \in C(T)$.

2. The Main Results

The following global unicity result on one-sided L_1-approximation is an immediate consequence of Theorem 1.

THEOREM 2. *For an n-dimensional subspace G of C(T) which contains a strictly positive function the following statements are equivalent*:

(1) *Every $f \in C(T)$ has a unique best one-sided L_1-approximation from G.*

(2) *Every $f \in C(T)$ has a strongly unique best one-sided L_1-approximation from G.*

(3) *There do not exist points $t_1, ..., t_{n-1} \in T$ and real numbers $a_1, ..., a_{n-1} \geq 0$ such that*

$$\int_T g \, d\mu = \sum_{i=1}^{n-1} a_i g(t_i) \qquad \text{for all } g \in G$$

(*i.e., there does not exist a quadrature formula of at most $n-1$ points and with positive weights which is exact for all $g \in G$*).

We now give a theorem on the existence of quadrature formulae as in Theorem 2 which will be proved later.

THEOREM 3. *Let $n \geq 2$, T have at most $n-1$ components, and G be an n-dimensional subspace of C(T) which contains a strictly positive function. Then there exists a quadrature formula of at most $n-1$ points and with positive weights which is exact for all $g \in G$.*

Combining Theorem 2 and Theorem 3 we immediately obtain the following general non-unicity result on one-sided L_1-approximation.

COROLLARY 4. *Let $n \geq 2$, T have at most $n-1$ components, and G be an n-dimensional subspace of C(T) which contains a strictly positive function. Then there exists a function in C(T) which has more than one best one-sided L_1-approximation from G.*

Remark 5. (1) Corollary 4 was proved in the special case when $T = [a, b]$ and under the additional assumption that G is a Haar subspace by DeVore [2], that G is a subspace of splines by Pinkus [9], and that G is a weak Chebyshev subspace (i.e., each function $g \in G$ has at most $n-1$ sign changes) by Strauss [11]. Note that Haar subspaces and subspaces of splines are weak Chebyshev.

(2) Theorem 2 shows that if G is a one-dimensional subspace of $C(T)$ which contains a strictly positive function, then every $f \in C(T)$ has a strongly unique best one-sided L_1-approximation from G.

(3) Note that Corollary 4 is also valid for functions in several variables, e.g., for subspaces of polynomials or splines in several variables.

In the following we will give a proof of Theorem 3. To do this we need the following well-known results (see, e.g., Brosowski [1] and Hettich and

Zencke [4]). For a subset A of $\mathbb{R}^n$ we denote by cone(A) the convex cone generated by A.

LEMMA 6 (Farkas). *Let vectors* $z_0, z_1, ..., z_m \in \mathbb{R}^n$ *be given. Then the following statements are equivalent*:

(1) $z_0 \in \text{cone}(\{z_i : i = 1, ..., m\})$.

(2) *There does not exist a vector* $y \in \mathbb{R}^n$ *such that*

$$\langle z_0, y \rangle < 0$$

and

$$\langle z_i, y \rangle \geqslant 0, \qquad i = 1, ..., m.$$

For a subset A of $\mathbb{R}^n$ *we denote by* conv(A) *the convex hull of A*.

LEMMA 7. *For a compact subset Z of* $\mathbb{R}^n$ *the following statements are equivalent*:

(1) $0 \in \text{conv}(Z)$.

(2) *There does not exist a vector* $y \in \mathbb{R}^n$ *such that*

$$\langle z, y \rangle > 0 \qquad \text{for all } z \in Z.$$

Proof of Theorem 3. Since G contains a strictly positive function, we may choose a basis $\{\tilde{g}_1, ..., \tilde{g}_n\}$ of G such that

$$\tilde{g}_n(t) > 0 \qquad \text{for all } t \in T.$$

Since μ is a strictly positive measure, we have $\int_T \tilde{g}_n \, d\mu > 0$. Therefore, there exist real numbers $b_1, ..., b_{n-1}$ such that

$$\int_T (\tilde{g}_i + b_i \tilde{g}_n) \, d\mu = 0, \qquad i = 1, ..., n-1.$$

We set $g_i = \tilde{g}_i + b_i \tilde{g}_n$, $i = 1, ..., n-1$, and $g_n = \tilde{g}_n$. Then $\{g_1, ..., g_n\}$ is a basis of G such that

(1) $\int_T g_i \, d\mu = 0$, $i = 1, ..., n-1$.

Then condition (3) in Theorem 2 says that

(2) there do not exist points $t_1, ..., t_{n-1} \in T$ and real numbers $a_1, ..., a_{n-1} \geqslant 0$ such that

$$\int_T g_j \, d\mu = \sum_{i=1}^{n-1} a_i g_j(t_i), \qquad j = 1, ..., n.$$

We set $x=(0,\ldots,0,\int_T g_n\,d\mu)$ and $G(t)=(g_1(t),\ldots,g_n(t))$ for all $t\in T$. Then condition (2) is equivalent to the following condition:

(3) there do not exist points $t_1,\ldots,t_{n-1}\in T$ such that

$$x\in\operatorname{cone}(\{G(t_i)\colon i=1,\ldots,n-1\}).$$

By Lemma 6 condition (3) is equivalent to the following condition:

(4) for all points $t_1,\ldots,t_{n-1}\in T$ there exist a vector $y\in\mathbb{R}^n$ such that

$$\langle x,y\rangle<0$$

and

$$\langle G(t_i),y\rangle\geqslant 0,\qquad i=1,\ldots,n-1.$$

We set $\tilde{G}=\operatorname{span}\{g_1,\ldots,g_{n-1}\}$. Then condition (4) is equivalent to the following condition:

(5) for all points $t_1,\ldots,t_{n-1}\in T$ there exists a function $\tilde{g}\in\tilde{G}$ such that

$$\tilde{g}(t_i)>0,\qquad i=1,\ldots,n-1.$$

In fact, if (5) holds, then there exist real numbers $y_1,\ldots,y_{n-1}$ such that

$$\sum_{j=1}^{n-1} y_j g_j(t_i)>0,\qquad i=1,\ldots,n-1.$$

Since $g_n(t)>0$ for all $t\in T$, we may choose a sufficiently small negative number y_n such that

$$\sum_{j=1}^{n} y_j g_j(t_i)>0,\qquad i=1,\ldots,n-1.$$

We set $y=(y_1,\ldots,y_n)$. Since $y_n\int_T g_n\,d\mu<0$, we have $\langle x,y\rangle>0$ which shows that (4) holds. The converse implication is obvious. To complete the proof we have to show that (5) fails. We set

$$\tilde{G}(t)=(g_1(t),\ldots,g_{n-1}(t))\qquad\text{for all } t\in T.$$

Since μ is a strictly positive measure, if follows from (1) that there does not exist a vector $\tilde{y}\in\mathbb{R}^{n-1}$ such that

$$\langle\tilde{G}(t),\tilde{y}\rangle>0\qquad\text{for all } t\in T.$$

Since T is compact and the function

$$t \to (g_1(t),..., g_{n-1}(t))$$

is continuous on T, $\{\tilde{G}(t): t \in T\}$ is a compact subset of $\mathbb{R}^{n-1}$. Therefore by Lemma 7 we have

$$0 \in \text{conv}(\{G(t): t \in T\}).$$

Moreover, since T has at most $n-1$ components, the set $\{\tilde{G}(t): t \in T\}$ has at most $n-1$ components. Therefore by Egglestone [3, p. 35] there exist points $t_1,..., t_{n-1} \in T$ such that

$$0 \in \text{conv}(\{\tilde{G}(t_i): i = 1,..., n-1\}).$$

Again by Lemma 7 there does not exist a function $\tilde{g} \in \tilde{G}$ such that

$$\tilde{g}(t_i) > 0, \qquad i = 1,..., n-1,$$

which shows that (5) fails. This proves Theorem 3.

In the following we will show that Theorem 3 and Corollary 4 are not true if T has more than $n-1$ components.

We first note that if an n-dimensional subspace G of $C(T)$ contains a strictly positive function, then there exists a basis $\{g_1,..., g_n\}$ of G such that

$$\int_T g_i \, d\mu = 0, \qquad i = 1,..., n-1,$$

and

$$g_n(t) > 0 \qquad \text{for all } t \in T.$$

Therefore, in this case we can always argue with such a basis. Then the proof of Theorem 3 yields the following result.

THEOREM 8. *Let $n \geq 2$, G be an n-dimensional subspace of $C(T)$, and $\{g_1,..., g_n\}$ be a basis of G such that*

$$\int_T g_i \, d\mu = 0, \qquad i = 1,..., n-1,$$

and

$$g_n(t) > 0 \qquad \textit{for all } t \in T.$$

Then the statements (1)–(3) *in Theorem* 2 *are equivalent to the following statement*:

(4) *For all points* $t_1, ..., t_{n-1} \in T$ *there exists a function* $g \in \text{span}\{g_1, ..., g_{n-1}\}$ *such that*

$$g(t_i) > 0, \qquad i = 1, ..., n-1.$$

If T has more than $n-1$ components, then it is easy to construct n-dimensional subspaces G of $C(T)$ for which condition (4) in Theorem 8 holds.

EXAMPLE 9. (1) Let $T = [-2, -1] \cup [1, 2]$ and $G = \text{span}\{g_1, g_2\}$ be a two-dimensional subspace of $C(T)$, defined by $g_1(t) = t$ for all $t \in T$ and $g_2 = 1$. Then condition (4) in Theorem 8 is satisfied. In particular all best approximations from G are strongly unique and G does not admit a quadrature formula as in Theorem 2.

(2) The space $l_1^m = \{(x_1, ..., x_m): x_i \text{ real}, i = 1, ..., m\}$ endowed with the norm

$$\|(x_1, ..., x_m)\|_1 = \sum_{i=1}^{m} |x_i|$$

is a space of type $C(T)$ endowed with the L_1-norm, where $T = \{1, ..., m\}$.

Let $G = \text{span}\{g_1, g_2\}$ be a two-dimensional subspace of l_1^m such that $g_1 = (x_1, ..., x_m)$ has the property that $x_i \neq 0$, $i = 1, ..., m$, and $\sum_{i=1}^{m} x_i = 0$ and $g_2 = (1, ..., 1)$.Then also condition (4) in Theorem 8 is satisfied.

Remark 10. (1) For further unicity results on one-sided L_1-approximation in $C[a, b]$ see, e.g., Sommer and Strauss [10] and Nürnberger, Schumaker, Sommer, and Strauss [8].

(2) For further unicity results on semi-infinite optimization see, e.g., Brosowski [1], Hettich and Zencke [4], and Nürnberger [5].

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